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STABILIZATION OF HOLONOMIC CONTROLLED SYSTEMS NEAR A POSITION OF EQUILIBRIUM†

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THIS PAPER is a continuation of the study of various laws of positional control of large dynamic systems [1–3], and deals with a holonomic controlled system near its position of equilibrium. The necessary and sufficient conditions for the existence of a control ensuring asymptotic stability of the system as a whole are obtained. A structure of the control, which is the simplest in a certain sense, which solves the problem in question, is given.

Let M, C, P, G be the matrices of mass, dissipative forces, potential energy and control, respectively, q the vector of generalized coordinates, and u the control. C and P are non-negative definite matrices, and M is a positive definite matrix. The motion of a holonomic system near the position of equilibrium is described by the equations [4]

$$Mq^{\cdot\cdot} + Cq^{\cdot} + Pq = Gu, \quad q \in R^n, \quad u \in R^m \tag{1}$$

Linear controlled systems of general type were studied in sufficient detail in [4, 5], and corresponding methods for obtaining a control for solving two-point boundary value problem were developed. If the system is of large dimensions the construction of positional control taking the system to a prescribed position is difficult. Therefore, regulators are often used which ensure the asymptotic stability of the dynamic system in the required position [5]. Suppose the system in question is of large dimensions, and it is required to construct a regulator which depends on the minimum number of generalized coordinates. Below we obtain the necessary and sufficient conditions determining the control matrix of such a regulator and the corresponding control is given.

We shall call the subspace L on which the non-negative definite form vanishes, the null subspace. We will denote by L_1 and L_2 the null subspaces of quadratic forms q^TCq and q^TPq (1) respectively.

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Theorem. Let us assume that $L_2 \subset L_1$ and $\dim L_2 = p$, and let the system of equations

$$\lambda^2 M\xi + P\xi = 0 \text{ and } C\xi = 0 \tag{2}$$

have no trivial general solutions for all $\lambda \neq 0$, including complex values. Then the following relation will represent the necessary and sufficient condition for the existence of a control ensuring the asymptotic stability as a whole:

$$\operatorname{rank} \| \xi^1, \xi^2, \ldots, \xi^p \|^T G = p \tag{3}$$

where ξ^i $(i = 1, \ldots, p)$ is an arbitrary set of linearly independent vectors forming the basis of the subspace L_2 in R^n.

Proof. Using the non-singular transformation $q = Sx$, we can reduce system (1) to normal coordinates

$$x_i^{\cdot\cdot} = (S^T Gu)_i, \quad i = 1, \ldots, p \tag{4}$$

$$x_i^{\cdot\cdot} + \sum_{j=p+1}^{n} (S^T CS)_{ii} x_j^{\cdot} + \omega_i^2 x_i = (S^T Gu)_i \quad i = p+1, \ldots, n, \quad \omega_i^2 = (S^T PS)_{ii} \tag{5}$$

By virtue of the assumption that $L_2 \subset L_1$, Eqs (4) contain no elements of the matrices C and P. Let us denote by S_0 the submatrix of the matrix S, formed by its first p columns. It follows from (4) that the necessary and sufficient condition for complete controllability of the system (4) is that the following relation holds:

$$\operatorname{rank} S_0^T G = p \tag{6}$$

Let $S_0^1, S_0^2, \ldots, S_0^p$ be the vectors forming the columns of the matrix S_0. The set of all linearly independent vectors forming the basis of L_2 and R^n, is given by the relation

$$\| \xi^1, \xi^2, \ldots, \xi^p \|^T = R \| S_0^1, S_0^2, \ldots, S_0^p \|^T$$

where R is an arbitrary, non-degenerate matrix of dimensions $p \times p$. Therefore relations (6) and (3) are equivalent.

If relation (3) is violated, then there is no control ensuring the asymptotic stability of system (4), and hence of the whole system (4), (5). Let relation (3) hold. We will denote by y the vector $(x_1, \ldots, x_p, x_1^{\cdot}, \ldots, x_p^{\cdot})^T$ and assume that the control $u(y)$ ensuring the asymptotic stability of system (4) has the following properties:

$$u(y(t)) \to 0 \quad \text{as} \quad t \to \infty \tag{7}$$

where $y(t)$ is the solution of system (4) corresponding to the control $u(y)$, and for any bounded region D of the phase space $y \in R^2p$ there exists a constant C_0 depending on the region and such, that

$$| u(y) | \leqslant C_0 \text{ when } y \in D \tag{8}$$

Let us substitute the control $u(y)$ into system (5), (5). We shall show that the control ensures the asymptotic stability of the complete system as a whole. System (4) is asymptotically stable by virtue of the choice of $u(y)$. We shall show that system (5) is also asymptotically stable.

Let $z = (x_{p+1}, \ldots, x_n, x_{p+1}^{\cdot}, \ldots, x_n^{\cdot})^T$ and let A be the matrix of phase coordinates of the system (5), reduced to normal form.

We shall show that all eigenvalues λ_k of the matrix A satisfy the condition

$$\operatorname{Re} \lambda_k < 0 \tag{9}$$

We shall seek the solution of the homogeneous system (4), (5) in the form $x = \eta e^{\lambda t}$, where η is a vector. Substituting this expression into the homogeneous system (4), (5) and dividing by $e^{\lambda t}$, we obtain

$$\lambda^2 \eta + \lambda S^T CS\eta + S^T PS \eta = 0 \tag{10}$$

A non-trivial solution of this system exists provided that λ satisfies the equation

$$\det (\lambda^2 E + \lambda S^T CS + S^T PS) = 0$$

From Eqs (10) and the assumption that $L_2 \subset L_1$ it follows that p eigen vectors $S\eta_i$ of the operator P correspond to $2p$ zero roots $\lambda_i = 0$. The vectors form the basis of the subspace L_2 in R^n, and the first p equations of system (4), (5) correspond to zero values of λ_i.

Let $\lambda_k \neq 0$ and η_k be the corresponding eigen vector satisfying system (10). We multiply $\eta_k{}^*$ from the left by the vector (10) and introduce the notation

$$a_1 = \eta^*\eta, \quad a_2 = \eta^* S^T C S \eta, \quad a_3 = \eta^* S^T P S \eta$$

We obtain $\lambda_k = (a_2 + (a_2{}^2 - 4a_1 a_3)^{1/2})/(2a_1)$. By virtue of the condition $\lambda_k \neq 0$ we have $S\eta_k \notin L_2$ and $a_1 > 0$, $a_3 > 0$. Therefore the necessary and sufficient condition for the inequality $\operatorname{Re}\lambda_k < 0$ to hold is that condition $a_2 > 0$ holds. The equality $a_2 = 0$ is possible if and only if the eigen vectors satisfying Eq. (10) make the quadratic form $\eta^* S^T C S \eta$ equal to zero. This however contradicts condition (2) of the theorem. Theorem $a_2 > 0$ and $\operatorname{Re}\lambda_k < 0$, which it was required to prove.

Any solution of Eq. (5) written in normal form, with control $u(y)$ is

$$z(t) = \exp(At)\, z^0 + I(t) \tag{11}$$

$$I(t) = \int_0^t \exp(A(t-\tau))\, f(\tau)\, d\tau \qquad (z^0 = z(0))$$

where $f(t)$ is the corresponding inhomogeneous part obtained when $u(y(t))$ is substituted into the right-hand side of (5).

In relation (11) the first term tends to zero as t increases by virtue of condition (9). In order to estimate the second term we use the inequality

$$\| \exp(At) \| \leqslant C \exp(\alpha t), \quad t \geqslant 0 \tag{12}$$

where $\alpha = \varepsilon + \max_k \operatorname{Re}\lambda_k < 0$, $\varepsilon > 0$, $k = 1, \ldots, 2(n-p)$, $C = \text{const}$.

We shall show that the second term on the right-hand side of relation (11) also tends to zero. According to estimate (12) we have

$$|I(t)| \leqslant C \int_0^t \exp(\alpha(t-\tau))\, |f(\tau)|\, d\tau$$

Let us write the integral on the right-hand side in the form of a sum

$$\exp(\alpha(t-T)) \int_0^T \exp(\alpha(T-\tau))\, |f(\tau)|\, d\tau + \int_T^t \exp(\alpha(t-\tau))\, |f(\tau)|\, d\tau$$

We shall choose T such, that when $\tau > T$ the inequality $|f(\tau)| \leqslant \delta$ holds and use property (8) of the control $u(y(t))$. The previous relation is not greater than

$$\alpha^{-1} [C_0 \| S^T G \| \exp(\alpha t)(1 - \exp(-\alpha T)) + \delta(\exp(\alpha(t-T)) - 1)]$$

The first term within the square brackets tends to zero as t increases by virtue of the fact that $\alpha < 0$. The second term can also be made as small as desired when δ is sufficiently small. Therefore the limit of the right-hand side of relation (11) is zero as $t \to \infty$, which it was required to prove.

Note. Condition (2) of the theorem is satisfied in advance when $L_1 = L_2$. In this case there exists a control ensuring the asymptotic stability of system (1) for any matrix C corresponding to this equality. Suppose now that the matrix C is such that condition $L_1 = L_2$ is violated and we have the inclusion $L_1 \subset L_2$. In this case the law of control will depend, generally speaking, on the dissipative forces.

Consider, as an example, the system

$$x_1^{\cdot\cdot} = u - \nu(0{,}99 x_1^{\cdot} + x_2^{\cdot})$$
$$x_2^{\cdot\cdot} = -0{,}1 x_2 - 0{,}3 x_2^{\cdot} - 10u - \nu(x_1^{\cdot} + 1{,}2 x_2^{\cdot})$$

When $\nu = 0$, the relation $L_1 = L_2$ holds and the control $u = -x_1 - 0.01\, x_1$ will ensure the asymptotic stability of the system. When $\nu = 1$, we have $L_1 \subset L_2$ and the control will render the system unstable. This can be confirmed by considering the corresponding characteristic polynomial. Therefore the law of control ensuring the asymptotic stability of a mechanical system in a vacuum will not necessarily guarantee it in a viscous medium.

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THE EQUILIBRIUM OF A PARABOLIC–LOGARITHMIC SHELL OF REVOLUTION†

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An exact general analytic solution is constructed for static membrane equations of equilibrium, in a complex form, for a parabolic–logarithmic shell of revolution with variable external load.

1. BASIC FORMULAS

STATIC momentless (mean brake) equilibrium of the middle surface of an elastic shell of revolution is described, in geographic coordinates z, θ, by the following system of partial differential equations:

$$\begin{gathered}\frac{\partial}{\partial z}(rT_1) - r'T_2 + t\frac{\partial S}{\partial \theta} + rtX_1 = 0 \\ t\frac{\partial T_2}{\partial \theta} + r\frac{\partial S}{\partial z} + 2r'S + rtX_2 = 0 \\ t^2T_2 - rr''T_1 + rt^3Z = 0 \quad (t = \sqrt{1 + r'^2})\end{gathered} \tag{1.1}$$

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